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Conformal covariantization of Moyal–Lax operators

Ming-Hsien Tu[1,3], **Niann-Chern Lee**[2] **and Yu-Tung Chen**[1]

[1] Department of Physics, National Chung Cheng University, Minghsiung, Chiayi 621, Taiwan, Republic of China
[2] General Education Centre, National Chinyi Institute of Technology, Taiping, Taichung 411, Taiwan, Republic of China

E-mail: phymhtu@ccunix.ccu.edu.tw

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Abstract

A covariant approach to the conformal property associated with Moyal–Lax operators is given. By identifying the conformal covariance with the second Gelfand–Dickey flow, we covariantize Moyal–Lax operators to construct the primary fields of one-parameter deformation of classical W-algebras.

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1. Introduction

Recently, there has been a great deal of interest in studying the Moyal deformation of the KdV equations in variant ways, such as Lax and/or Hamiltonian formulations [1–5], the zero-curvature formulation [6], Bäcklund transformation [7] and classical Virasoro and W-algebras [3, 4]. In these formulations, the ordinary (pseudo-) differential Lax operator $L = \sum_i u_i(x)\partial^i$ is replaced by (pseudo-) differential symbols $M(x, p)$, the formal Laurent series in p, which obey a noncommutative but associative algebra with respect to the $\star$-product [8] defined by

$$M(x,p)\star N(x,p) = \sum_{s=0}^{\infty}\frac{\theta^s}{s!}\sum_{j=0}^{s}(-1)^j\binom{s}{j}(\partial_x^j\partial_p^{s-j}M)(\partial_x^{s-j}\partial_p^jN), \tag{1.1}$$

where θ is a dimensionless parameter characterizing the strength of the deformation. On the other hand, by (1.1), the ordinary commutator is thus taken over by the Moyal bracket [9]

$$\{M(x,p), N(x,p)\}_\theta = \frac{M\star N - N\star M}{2\theta}, \tag{1.2}$$

that possesses the anti-symmetry, bi-linearity and Jacobi identity. The Moyal bracket (1.2) can be viewed as the higher-order derivative (or dispersive) generalization of the canonical Poisson bracket since it recovers the canonical Poisson bracket in the limit $\theta \to 0$, namely,

[3] Author to whom any correspondence should be addressed.

$\lim_{\theta\to 0}\{M, N\}_\theta = \partial_p M \partial_x N - \partial_x M \partial_p N$. It turns out that the Moyal formulation of Lax equations reduces to dispersionless Lax equations [10–14] in this limit.

In our previous work [3] we studied the W-algebraic structure associated with the Moyal formulation of the KdV equations. We worked out the Poisson brackets of the second Gelfand–Dickey (GD) structure [15,16] defined by the $\star$-product and obtained a one-parameter deformation of the classical W_n-algebra including a Virasoro subalgebra with central charge $\theta^2(n^3-n)/3$. In this work, we would like to investigate further the W-algebraic structure from the point of view of conformal covariance. We shall follow the approach developed by Di Francesco, Itzykson and Zuber (DIZ) [17] to covariantize the Moyal–Lax operators [1,4] and identify the underlying primary fields in a systematic way.

This paper is organized as follows. In section 2, we recall the Moyal–Lax formulation of the KdV equations using psuedo-differential symbols with respect to the $\star$-product. We then introduce the second GD structure defined by the Moyal bracket and show that it indeed provides the Hamiltonian structure for the Moyal-type Lax equations. In section 3, the diffeomorphism (S^1) is defined and the conformal transformation of the Moyal–Lax operators is investigated. Then, in section 4, we show that the infinitesimal diffeomorphism flow defined by conformal covariance is equivalent to that of the Hamiltonian flow defined by the second GD structure. This enables us to define the primary fields of the diffeomorphism. Following DIZ, in section 5, we systematically covariantize the Moyal–Lax operator to decompose the coefficient functions of the Lax operator into the conformal primary fields which satisfy a one-parameter deformation of the classical W_n-algebra including a Virasoro subalgebra. In section 6 the covariantization is generalized to psuedo-differential symbols to construct additional primary fields. Section 7 is devoted to the conclusions and discussion.

2. Lax equations and Hamiltonian structures

For the differential symbol $L = p^n + \sum_{i=1}^n u_i \star p^{n-i}$ with coefficients u_i depending on an infinite set of variables $x \equiv t_1, t_2, t_3, \ldots$ one can define the Lax equations [1,4]

$$\begin{aligned}\frac{\partial L}{\partial t_k} &= \{(L^{1/n}\star)^k_+, L\}_\theta, \qquad (L^{1/n}\star)^k_+ = (\underbrace{L^{1/n}\star L^{1/n}\star\cdots\star L^{1/n}}_{k})_+,\\ &= \{L, (L^{1/n}\star)^k_-\}_\theta,\end{aligned} \tag{2.1}$$

where $L^{1/n} = p + \sum_{i=0}^\infty a_i \star p^{-i}$ is the nth root of L in such a way that $L = (L^{1/n}\star)^n$ and $(A)_{+/-}$ refer to the non-negative/negative powers in p of the pseudo-differential symbol A. Note that the evolution equation for u_1 is trivial since the highest order in p on the right-hand side of the Lax equations (2.1) is $n-2$ due to the definition of the Moyal bracket, and hence one can drop u_1 from the Lax formulation. However, we shall see that this is not the case for the Hamiltonian formulation.

Next let us formulate the Lax equations (2.1) in terms of Hamiltonian structure. For the functionals $F[L]$ and $G[L]$ we define the second GD bracket [16] with respect to the $\star$-product as

$$\{F, G\}_2 = \mathrm{tr}\,[J^{(2)}(d_L F)\star d_L G] = \int \mathrm{res}\,[J^{(2)}(d_L F)\star d_L G], \tag{2.2}$$

where $\mathrm{res}\,(A) = a_{-1}$ and $\mathrm{tr}(A) = \int \mathrm{res}\,(A)$ denote the residue and trace of $A = \sum_i a_i \star p^i$, and $J^{(2)}$ is the Adler map [18] defined by

$$\begin{aligned}J^{(2)}(d_L F) &= \{L, d_L F\}_{\theta+}\star L - \{L, (d_L F\star L)_+\}_\theta,\\ &= \{L, (d_L F\star L)_-\}_\theta - \{L, d_L F\}_{\theta-}\star L,\end{aligned} \tag{2.3}$$

with $d_L F \equiv \delta F/\delta L = \sum_{i=1}^{n} p^{-n+i-1} \star \delta F/\delta u_i$. The bracket defined by $J^{(2)}$ is indeed Hamiltonian since $\{F, G\}_2 = -\{G, F\}_2$ due to the cyclic property of the trace and the Jacobi identity can be verified [7] by the Kupershmidt–Wilson (KW) theorem [19]. Form (2.3) $J^{(2)}(X)$ is linear in X and has order at most $n-1$. One can use the standard Dirac procedure [17] to get rid of u_1 so that

$$\hat{J}^{(2)}(X) = \{L, X\}_{\theta+} \star L - \{L, (X \star L)_+\}_\theta + \frac{1}{n}\left\{L, \int^x \operatorname{res}\{L, X\}_\theta\right\}_\theta \tag{2.4}$$

or, in components, $\hat{J}^{(2)}(X) = \sum_{i,j=2}^{n} (\hat{J}_{ij}^{(2)} \cdot x_j) \star p^{n-i}$, where $\hat{J}_{ij}^{(2)}$ are differential operators, and hence the reduced Poisson brackets for u_i can be expressed as $\{u_i(x), u_j(y)\}_2^D = \hat{J}_{ij}^{(2)} \cdot \delta(x-y)$. From the reduced GD brackets (2.4) the Hamiltonian flows can be expressed as

$$\frac{\partial L}{\partial t_k} = \{L, H_k\}_2^D = \hat{J}^{(2)}(d_L H_k), \tag{2.5}$$

where the Hamiltonians H_k are defined by

$$H_k = \frac{n}{k} \int \operatorname{res}(L^{1/n}\star)^k. \tag{2.6}$$

Using (2.6) and the fact $d_L H_k = (L^{1/n}\star)_-^{k-n} \bmod p^{-n}$ it is straightforward to show that the Hamiltonian flows (2.5) are equivalent to the Lax equations (2.1).

3. Diff (S^1) and conformal covariance

A function $f(x)$ is a primary field with conformal weight h if under the diffeomorphism $x \to t(x)$ it transforms as

$$f(x) \to \tilde{f}(t) = \phi^{-h} f(x) = \phi^{-h} \star f(x), \tag{3.1}$$

where $\phi(x) \equiv \mathrm{d}t(x)/\mathrm{d}x$. We denote by $\mathcal{F}_h$ the space of functions with weight h (or spin-h primary fields). For a covariant operator $\Delta(x, p)$ that maps $\mathcal{F}_h$ to $\mathcal{F}_l$, it transforms according to

$$\tilde{\Delta}(t, p_t) = \phi^{-l} \star \Delta(x, p) \star \phi^h, \tag{3.2}$$

where $p_t = \phi^{-1} \star p$ is the conjugate momentum of t with respect to the Moyal bracket, i.e. $\{p_t, t\}_\theta = 1$, and has an inverse $p_t^{-1} = p^{-1} \star \phi$.

Let us treat the Lax operator $L_n(x, p) = p^n + u_2(x) \star p^{n-2} + \cdots + u_n(x)$ as a covariant operator such that $L_n(x, p) : \mathcal{F}_h \to \mathcal{F}_l$. The corresponding weights h and l have to be determined from the transformation law:

$$\begin{aligned}\tilde{L}_n(t, p_t) &= \phi^{-l} \star L_n(x) \star \phi^h, \\ &= (p_t\star)^n + \tilde{u}_2(t) \star (p_t\star)^{n-2} + \cdots + \tilde{u}_n(t).\end{aligned} \tag{3.3}$$

We note that $p_t = \phi^{-1} \star p = (\sqrt{\phi})^{-1} \star \phi^{-1} p \star \sqrt{\phi}$, which, by induction, gives

$$(p_t\star)^k = \frac{1}{\sqrt{\phi}} \star \left[\phi^{-k} p^k + \frac{\theta^2 f_k}{\phi^k} p^{k-2} + \cdots\right] \star \sqrt{\phi}, \tag{3.4}$$

with

$$f_k = -\frac{k(k-1)}{2}\left(\frac{\phi'}{\phi}\right)^2 - \frac{k(k-1)(k-2)}{6}\frac{\phi''}{\phi}.$$

Substituting (3.4) into (3.3) we have $h = -(n-1)/2, \quad l = (n+1)/2$ and $u_2(x)$ transforms like an anomalous spin-2 primary field

$$\tilde{u}_2(t) = \phi^{-2} u_2(x) + \frac{\theta^2(n^3-n)}{3}\{\{x, t(x)\}\}, \tag{3.5}$$

where $\{\{x, t(x)\}\}$ is the Schwarzian derivative defined by

$$\{\{x, t(x)\}\} = \left(\frac{\mathrm{d}^3x/\mathrm{d}t^3}{\mathrm{d}x/\mathrm{d}t}\right) - \left(\frac{\mathrm{d}^2x/\mathrm{d}t^2}{\mathrm{d}x/\mathrm{d}t}\right)^2 = \frac{\phi''}{\phi^3} - \frac{3}{2}\left(\frac{\phi'}{\phi^2}\right)^2. \tag{3.6}$$

Equation (3.5) indicates that u_2 can be viewed as the generator of the classical Virasoro algebra with central charge $c_{n,\theta} = \theta^2(n^3 - n)/3$.

4. Virasoro flows as Hamiltonian flows

As shown in the previous section, is quite difficult to obtain the transformation laws for $u_{i>2}$ under the finite diffeomorphism. However it is possible to investigate the infinitesimal transformations of u_i. For an infinitesimal diffeomorphism $x \to t(x) \simeq x - \epsilon(x)$ we have $\phi(x) \simeq 1 - \epsilon'(x)$ and $p_t = p + \{p, \epsilon\}_\theta \star p$. In particular, it can be easily proved by induction that $(p_t\star)^i = p^i + \{p^i, \epsilon\}_\theta \star p$. Hence from (3.3) we have

$$\begin{aligned}\tilde{L}_n(t) &= \sum_i (u_i(x) - \epsilon(x)u_i'(x) + \delta_\epsilon u_i(x)) \star (p^i + \{p^i, \epsilon\}_\theta \star p),\\ &= L_n(x) + \{L_n(x), \epsilon(x)\}_\theta \star p - \epsilon(x) \star L_n'(x) + \delta_\epsilon L_n(x),\\ &= \left(1 + \frac{n+1}{2}\epsilon'(x)\right) \star L_n(x) \star \left(1 + \frac{n-1}{2}\epsilon'(x)\right),\\ &= L_n(x) + \frac{n+1}{2}\epsilon'(x) \star L_n(x) + \frac{n-1}{2}L_n(x) \star \epsilon'(x),\end{aligned}$$

which leads to the infinitesimal change of the Lax operator

$$\begin{aligned}\delta_\epsilon L_n(x) = {}& \frac{n+1}{2}\epsilon'(x) \star L_n(x) + \frac{n-1}{2}L_n(x) \star \epsilon'(x)\\ & - \{L_n(x), \epsilon(x)\}_\theta \star p + \epsilon(x) \star \{p, L_n(x)\}_\theta.\end{aligned} \tag{4.1}$$

Next let us consider the Hamiltonian flow generated by the Hamiltonian $H = \int \epsilon(x)u_2(x)\,\mathrm{d}x$. From the second GD structure (2.4) and Hamiltonian flow (2.5) we have

$$\begin{aligned}\delta^{GD}L_n(x) = {}& \{L_n(x), X\}_\theta \star L_n(x) - \{L_n(x), (X \star L_n(x))_+\}_\theta\\ & + \frac{1}{n}\left\{L_n(x), \int^x \mathrm{res}\,\{L_n(x), X\}_\theta\right\}_\theta,\end{aligned}$$

where $X \equiv \delta H/\delta L = p^{-n+1} \star \epsilon(x)$. Simple algebra shows that

$$\begin{aligned}&(L_n \star X)_+ = p \star \epsilon,\\ &(X \star L_n)_+ = \epsilon \star p - 2\theta(n-1)\epsilon',\\ &\frac{1}{n}\left\{L_n, \int^x \mathrm{res}\,\{L_n, X\}_\theta\right\}_\theta = -\frac{n-1}{2}(L_n \star \epsilon' - \epsilon' \star L_n),\end{aligned}$$

which implies

$$\begin{aligned}\delta^{GD}L_n &= \frac{1}{2\theta}[p \star \epsilon \star L_n - L_n \star (\epsilon \star p - 2\theta(n-1)\epsilon')] - \frac{n-1}{2}(L_n \star \epsilon' - \epsilon' \star L_n),\\ &= \delta_\epsilon L_n,\end{aligned}$$

as desired. Comparing the two sides of (4.1) we obtain the infinitesimal variations of u_k $(2 \leqslant k \leqslant n)$ as

$$\begin{aligned}\delta_\epsilon u_k = {}& u_k'\epsilon + ku_k\epsilon' + \frac{(2\theta)^k(k-1)}{2}\binom{n+1}{k+1}\epsilon^{(k+1)}\\ & + \sum_{i=2}^{k-1}(2\theta)^{k-i}\left[\frac{n-1}{2}\binom{n-i}{k-i} - \binom{n-i}{k-i+1}\right]u_i\epsilon^{(k-i+1)},\end{aligned} \tag{4.2}$$

where $\binom{n}{m}$ are the standard binomial coefficients with $0 \leqslant m \leqslant n$. Let us list the first few δu_k:

$$\begin{aligned}
\delta_\epsilon u_2 &= u_2'\epsilon + 2u_2\epsilon' + \frac{\theta^2(n^3-n)}{3}\epsilon''', \\
\delta_\epsilon u_3 &= u_3'\epsilon + 3u_3\epsilon' + 2\theta(n-2)u_2\epsilon'' + \frac{\theta^3(n^3-n)(n-2)}{3}\epsilon^{(4)}, \\
\delta_\epsilon u_4 &= u_4'\epsilon + 4u_4\epsilon' + 3\theta(n-3)u_3\epsilon'' + \frac{\theta^2(n-2)(n-3)(n+5)}{3}u_2\epsilon''' \\
&\quad + \frac{\theta^4(n-2)(n-3)(n^3-n)}{5}\epsilon^{(5)}, \\
\delta_\epsilon u_5 &= u_5'\epsilon + 5u_5\epsilon' + 4\theta(n-4)u_4\epsilon'' + \frac{\theta^2(n-3)(n-4)(n+7)}{3}u_3\epsilon''' \\
&\quad + \frac{\theta^3(n-2)(n-3)(n-4)(n+3)}{3}u_2\epsilon^{(4)} \\
&\quad + \frac{4\theta^5(n-2)(n-3)(n-4)(n^3-n)}{45}\epsilon^{(6)},
\end{aligned} \tag{4.3}$$

etc. The first equation in (4.3) is just the infinitesimal version of (3.5), which, together with the Hamiltonian flow $\delta_\epsilon u_2(x) = \{u_2(x), H\}_2^D = \int \{u_2(x), u_2(y)\}_2^D \epsilon(y)\,\mathrm{d}y$ implies the classical Virasoro algebra

$$\{u_2(x), u_2(y)\}_2^D = [c_{n,\theta}\partial_x^3 + 2u_2\partial_x + u_2']\delta(x-y). \tag{4.4}$$

Furthermore, it has a simple interpretation of the other relations in (4.3). We can define a new variable $w_k = u_k + f(u_i)$, where $f(u_i)$ is a differential polynomial in $u_{i<k}$, such that w_k is a spin-k primary field with respect to the generator u_2, namely,

$$\{w_k(x), u_2(y)\}_2 = [kw_k\partial_x + w_k']\delta(x-y).$$

For instance, let $w_3 = u_3 + \alpha u_2'$; demanding the relation $\delta_\epsilon w_3 = \epsilon w_3' + 3w_3\epsilon'$, we obtain $\alpha = -\theta(n-2)$. On the other hand, let $w_4 = u_4 + \alpha u_3' + \beta u_2'' + \gamma u_2^2$; demanding the relation $\delta_\epsilon w_4 = \epsilon w_4' + 4w_4\epsilon'$ we have $\alpha = -\theta(n-3)$, $\beta = 2\theta^2(n-2)(n-3)/5$ and $\gamma = -(n-2)(n-3)(5n+7)/[10(n^3-n)]$.

In summary, we can identify the following primary fields:

$$\begin{aligned}
w_3 &= u_3 - \theta(n-2)u_2', \\
w_4 &= u_4 - \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)}u_2^2 - \theta(n-3)u_3' + \frac{2\theta^2(n-2)(n-3)}{5}u_2'', \\
w_5 &= u_5 - \theta(n-4)u_4' + \frac{3\theta^2(n-3)(n-4)}{7}u_3'' - \frac{2\theta^3(n-2)(n-3)(n-4)}{21}u_2''' \\
&\quad + \frac{(n-3)(n-4)(7n+13)}{7(n^3-n)}[\theta(n-2)u_2u_2' - u_2u_3],
\end{aligned} \tag{4.5}$$

etc. To construct the primary fields w_k for $k > 5$ we shall covariantize the Lax operator in a systematic way.

5. Covariantizing the Lax operators

For a series of changes of variable $v \to x \to t$, the Schwarzian derivative obeys the equation

$$\{\{v,t\}\} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \{\{v,x\}\} + \{\{x,t\}\}, \tag{5.1}$$

which, comparing with (3.5), shows that $u_2(x)$ transforms as $c_{n,\theta}\{\{v,x\}\}$. Define the variable $b(x) = \frac{\mathrm{d}^2v}{\mathrm{d}x^2}(\frac{\mathrm{d}v}{\mathrm{d}x})^{-1}$; it turns out that, for $n \neq -1, 0, 1$ and $\theta \neq 0$,

$$\frac{u_2(x)}{c_{n,\theta}} = \{\{v,x\}\} = b'(x) - \frac{1}{2}b^2(x), \tag{5.2}$$

with v being the coordinate where u_2 vanishes, i.e. $u_2(v) = 0$. It is easy to show that $b(x)$ transforms as an anomalous spin-1 primary field

$$\tilde{b}(t) = \frac{d^2 v}{dt^2}\left(\frac{dv}{dt}\right)^{-1} = \frac{dx}{dt}b(x) + \frac{d^2 x}{dt^2}\left(\frac{dx}{dt}\right)^{-1}. \tag{5.3}$$

The purpose of introducing $b(x)$ is to construct a covariant operator $D_k = p - 2\theta k b(x)$, which maps $\mathcal{F}_k$ to $\mathcal{F}_{k+1}$. Using D_k, the covariant operator $D_k^l : \mathcal{F}_k \to \mathcal{F}_{k+l}$ can be constructed as $D_k^l = D_{k+l-1} \star D_{k+l-2} \star \cdots \star D_k (l > 1)$.

Now, following the DIZ procedure, the Lax operator L_n can be decomposed into the sum of the covariant operators $\Delta_k^{(n)} : \mathcal{F}_{-\frac{n-1}{2}} \to \mathcal{F}_{\frac{n+1}{2}}$ as

$$L_n = \Delta_2^{(n)}(u_2) + \Delta_3^{(n)}(w_3, u_2) + \cdots + \Delta_n^{(n)}(w_n, u_2), \tag{5.4}$$

where

$$\Delta_2^{(n)} = D^n_{-\frac{n-1}{2}} = [p - \theta(n-1)b(x)] \star [p - \theta(n-3)b(x)] \star \cdots \star [p + \theta(n-1)b(x)],$$

$$\Delta_k^{(n)} = \sum_{l=0}^{n-k} \alpha_{k,l}^{(n)} (D_k^l \star w_k) \star D^{n-k-l}_{-\frac{n-1}{2}},$$

and the coefficients $\alpha_{k,l}^{(n)}$ are determined from the requirement that the Lax operator L_n depends on u_2 only through the relation (5.2). Therefore the function $b(x)$ is defined up to the condition $(\delta b)' - b\delta b = 0$ or, equivalently, $D_{k+1} \star \delta b = \delta b \star D_k$. In particular we have

$$\begin{aligned}
\delta_b D_k^l &= \sum_{i=1}^{l} D_{k+l-1} \star \cdots \star \delta_b D_{k+l-i} \star \cdots \star D_k, \\
&= \sum_{i=1}^{l} D_{k+l-1} \star \cdots \star [-2\theta(k+l-i)\delta b] \star \cdots \star D_k, \\
&= -\theta l(2k+l-1)\delta b \star D_k^{l-1}.
\end{aligned} \tag{5.5}$$

Hence $\delta_b L_n = 0$ implies

$$\delta_b D^n_{-\frac{n-1}{2}} + \sum_{k=3}^{n}\sum_{l=0}^{n-k}[\alpha_{k,l}^{(n)}(\delta_b D_k^l \star w_k) \star D^{n-k-l}_{-\frac{n-1}{2}} + \alpha_{k,l}^{(n)}(D_k^l \star w_k) \star \delta_b D^{n-k-l}_{-\frac{n-1}{2}}] = 0. \tag{5.6}$$

From (5.5) it is easy to show that the first term in (5.6) vanishes. For those terms in summation we obtain the recursive relation

$$\alpha_{k,l+1}^{(n)} = \frac{(k+l)(n-k-l)}{(2k+l)(l+1)}\alpha_{k,l}^{(n)}, \qquad k \geqslant 3$$

which together with the normalization condition $\alpha_{k,0}^{(n)} = 1$ yields

$$\alpha_{k,l}^{(n)} = \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}}.$$

Let us work out the first few terms for the decomposition (5.4). A straightforward computation yields

$$\begin{aligned}
(D_k \star w_k) &= 2\theta(w_k' - kbw_k), \\
(D_k^2 \star w_k) &= 4\theta^2[w_k'' - (2k+1)bw_k' + (k(k+1)b^2 - kb')w_k], \\
(D_k^3 \star w_k) &= 8\theta^3[w_k''' - 3(k+1)bw_k'' - (3k+1)b'w_k' + (3k^2+6k+2)b^2 w_k' \\
&\quad - k(k+1)(k+2)b^3 w_k + k(3k+4)bb'w_k - kb''w_k],
\end{aligned} \tag{5.7}$$

and

$$
\begin{aligned}
D^n_{-\frac{n-1}{2}} &= p^n + u_2 \star p^{n-2} + \theta(n-2)u_2' \star p^{n-3} \\
&\quad + \left[\frac{3\theta^2(n-2)(n-3)}{5}u_2'' + \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)}u_2^2\right] \star p^{n-4} \\
&\quad + \left[\frac{\theta(n-2)(n-3)(n-4)(5n+7)}{5(n^3-n)}u_2u_2' \right. \\
&\quad \left. + \frac{4\theta^3(n-2)(n-3)(n-4)}{15}u_2'''\right] \star p^{n-5} + \cdots, \\
D^{n-3}_{-\frac{n-1}{2}} &= p^{n-3} + 3\theta(n-3)b \star p^{n-4} \\
&\quad + \left[\frac{\theta^2(n-3)(n-4)(n+7)}{3}b' - \frac{(n-3)(n-4)(n-29)}{6}b^2\right] \star p^{n-5} + \cdots, \\
D^{n-4}_{-\frac{n-1}{2}} &= p^{n-4} + 4\theta(n-4)b \star p^{n-5} + \cdots, \\
D^{n-5}_{-\frac{n-1}{2}} &= p^{n-5} + \cdots.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Delta_2^{(n)}(u_2) &= D^n_{-\frac{n-1}{2}}, \\
\Delta_3^{(n)}(w_3, u_2) &= w_3 \star p^{n-3} + \theta(n-3)w_3' \star p^{n-4} \\
&\quad + \left[\frac{4\theta^2(n-3)(n-4)}{7}w_3'' + \frac{(n-3)(n-4)(7n+13)}{7(n^3-n)}u_2w_3\right] \star p^{n-5} + \cdots, \\
\Delta_4^{(n)}(w_4, u_2) &= w_4 \star p^{n-4} + \theta(n-4)w_4' \star p^{n-5} + \cdots, \\
\Delta_5^{(n)}(w_5, u_2) &= w_5 \star p^{n-5} + \cdots,
\end{aligned}
$$

which decomposes the coefficient functions u_i into the primary fields

$$
\begin{aligned}
u_2 &= w_2, \\
u_3 &= w_3 + \theta(n-2)u_2', \\
u_4 &= w_4 + \theta(n-3)w_3' + \frac{3\theta^2(n-2)(n-3)}{5}u_2'' + \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)}u_2^2, \\
u_5 &= w_5 + \theta(n-4)w_4' + \frac{4\theta^2(n-3)(n-4)}{7}w_3'' + \frac{(n-3)(n-4)(7n+13)}{7(n^3-n)}w_3u_2 \\
&\quad + \frac{\theta(n-2)(n-3)(n-4)(5n+7)}{5(n^3-n)}u_2u_2' + \frac{4\theta^3(n-2)(n-3)(n-4)}{15}u_2'''.
\end{aligned}
\tag{5.8}
$$

Inverting the above relation we recover the definition (4.5) of the primary fields.

6. Generalizations

In this section we would like to show that the conformal covariantization for the Lax operator (3.3) can be extended to a more general form

$$
\Lambda_n = p^n + u_2 \star p^{n-2} + \cdots + u_n + u_{n+1} \star p^{-1} + u_{n+2} \star p^{-2} + \cdots. \tag{6.1}
$$

It is not hard to show that, for the pseudo-differential symbol (6.1), the associated Hamiltonian structure is defined by the reduced Adler map (2.4) as well. Due to the fact that $(\Lambda_n)_+$ and

$(\Lambda_n)_-$ are transformed independently under (3.3), the infinitesimal change of u_k $(2 \leqslant k \leqslant n)$ is the same as (4.3), while that of u_{n+k} $(k \geqslant 1)$, governed by (4.1), yields

$$\delta u_{n+k} = u'_{n+k}\epsilon + (n+k)u_{n+k}\epsilon' + \sum_{i=1}^{k-1}(2\theta)^{k-i}\left[\frac{n-1}{2}\binom{-i}{k-i} - \binom{-i}{k-i+1}\right]u_{n+i}\epsilon^{(k-i+1)},$$

where $\binom{-n}{m} \equiv (-1)^m\binom{n+m-1}{m}$ with $n, m \geqslant 0$, from which the following primary fields can be defined:

$$\begin{aligned}
w_{n+1} &= u_{n+1},\\
w_{n+2} &= u_{n+2} + \theta u'_{n+1},\\
w_{n+3} &= u_{n+3} + 2\theta u'_{n+2} + \frac{2\theta^2(n+1)}{2n+3}u''_{n+1} - \frac{6(n+1)}{n(n-1)(2n+3)}u_2u_{n+1},\\
w_{n+4} &= u_{n+4} + 3\theta u'_{n+3} + \frac{6\theta^2(n+2)}{2n+5}u''_{n+2} + \frac{2\theta^3(n+1)}{2n+5}u'''_{n+1}\\
&\quad - \frac{6(3n+7)}{n(n-1)(2n+5)}u_2u_{n+2} - \frac{6(3n+7)}{n(n-1)(2n+5)}u_2u'_{n+1},
\end{aligned} \tag{6.2}$$

etc. To covariantize the negative part $(\Lambda_n)_-$ one can define the covariant operator $D_k^{-1} : \mathcal{F}_k \to \mathcal{F}_{k-1}$ as [20]

$$D_k^{-1} \equiv [D_{k-1}]^{-1} = p^{-1} + 2\theta(k-1)b \star p^{-2} + \cdots, \tag{6.3}$$

and thus

$$D_k^{-l} = [D_{k-l}^l]^{-1} = D_{k-l-1}^{-1} \star D_{k-l}^{-1} \cdots \star D_k^{-1}, \tag{6.4}$$

with a covariant property determined by that of D_{k-l}^l as

$$D_k^{-l}(t) = [D_{k-l}^l(t)]^{-1} = \phi^{l-k} \star D_k^{-l}(x) \star \phi^k.$$

Now let us decompose $(\Lambda_n)_-$ as

$$(\Lambda_n)_- = \sum_{l=1}^{\infty} \Delta_{n+k}^{(n)}(w_{n+k}, u_2), \tag{6.5}$$

where the covariant operator $\Delta_{n+k}^{(n)}(w_{n+k}, u_2)$ is linear in w_{n+k} and is defined by

$$\Delta_{n+k}^{(n)}(w_{n+k}, u_2) = \sum_{l=0}^{\infty} \beta_{n+k,l}^{(n)}(D_{n+k}^l \star w_{n+k}) \star D_{-\frac{n-1}{2}}^{-k-l}, \qquad k \geqslant 1.$$

The coefficients $\beta_{n+k,l}^{(n)}$ can be determined in a similar manner so that $(\Lambda_n)_-$ depends on u_2 only through (5.2). It turns out that

$$\beta_{n+k,l}^{(n)} = (-1)^l \frac{\binom{k+l-1}{l}\binom{n+k+l-1}{l}}{\binom{2n+2k+l-1}{l}}.$$

Following a similar procedure discussed in the previous section and comparing (6.1) with (6.5) we obtain

$$\begin{aligned}
u_{n+1} &= w_{n+1},\\
u_{n+2} &= w_{n+2} - \theta w'_{n+1},\\
u_{n+3} &= w_{n+3} - 2\theta w'_{n+2} + \frac{2\theta^2(n+2)}{2n+3}w''_{n+1} + \frac{6(n+1)}{n(n-1)(2n+3)}u_2w_{n+1},\\
u_{n+4} &= w_{n+4} - 3\theta w'_{n+3} + \frac{6\theta^2(n+3)}{(2n+5)}w''_{n+2} + \frac{6(3n+7)}{n(n-1)(2n+5)}u_2w_{n+2}\\
&\quad - \frac{2\theta^3(n+3)}{(2n+3)}w'''_{n+1} - \frac{18\theta(n+1)}{n(n-1)(2n+3)}(u_2w_{n+1})'.
\end{aligned} \tag{6.6}$$

Inverting the above equations yields (6.2) as expected.

7. Conclusion and discussions

We have discussed the covariance of the Moyal-type Lax operator under the diffeomorphism (S^1). By comparing the infinitesimal Diff (S^1) flow with the GD flow we have identified the primary fields with respect to the classical energy–momentum generator which obeys the classical Virasoro algebra with central charge $c_{n,\theta} = \theta^2(n^3 - n)/3$. We then follow the DIZ procedure to covariantize Moyal–Lax operators and identify the primary fields in a systematic way.

A few remarks are in order. First, the w_k shown above form a one-parameter deformation of the primary fields arising from the (pseudo-)differential Lax operator. In particular, the central charge $c_{n,\theta}$ can be used to characterize the dispersion effect since $\theta \to 0$ corresponds to the dispersionless limit of the Lax equation (2.1). Secondly, for $\theta = 1/2$ the primary fields w_k recover the standard result [17, 20], while for $\theta = 0$ (4.5) they do not directly reproduce those results in the dispersionless limit [13] in which the coefficient functions u_k are already primary fields with respect to u_2, the generator of the centreless Virasoro algebra. This is due to the fact that the parametrization (5.2) does not work for $\theta = 0$ and thus the associated conformal property should be traced back to the GD structure or infinitesimal transformation (4.2). Thirdly, in spite of covariantizing the Lax operator $L_n = p^n + \sum_{i=2}^{n} u_i \star p^{n-i}$, the conformal property associated with the Lax operator of the form

$$K_n = p^n + v_2 p^{n-2} + v_3 p^{n-3} + \cdots + v_n \tag{7.1}$$

has been investigated [3] as well. In fact, the Lax equations defined by K_n and L_n are equivalent up to the following isomorphism:

$$v_j = \sum_{i=1}^{j} (-\theta)^{j-i} \binom{n-i}{n-j} u_i^{(j-i)}, \tag{7.2}$$

which can be used to construct the primary fields associated with K_n. For instance, from (4.5) and (7.2), the first few primary fields can be expressed as

$$\begin{aligned} w_2 &= v_2, \\ w_3 &= v_3, \\ w_4 &= v_4 - \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)} v_2^2 - \frac{\theta^2(n-2)(n-3)}{10} v_2'', \end{aligned}$$

which are just those primary fields obtained in [3].

Finally, based on the algebra of pseudo-differential symbols with respect to the $\star$-product, it would be intriguing to carry out the covariant approach for reductions, truncations and even supersymmetrization [5] of the Lax operator (6.1) to construct the corresponding W-algebras. Work in these directions is now in progress.

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